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MULTIPLE BINARY FORMS WITH THE CLOSURE PROPERTY.

By Arthur B. Coble.[*]

Introduction.

Interesting accounts of the many investigations which have centered about the Poncelet polygons associated with the double binary forms of order (2, 2) with the closure property have been given by G. Loria[†] and H. S. White.[‡] The latter has called attention also to the existence of forms of order (3, 3) with a similar property.[§]

It is the purpose of this paper to discuss the conditions for closure of the general double binary form of order (k, κ) and to indicate certain possible extensions to multiple forms. It will appear that the closure property for a given form implies the existence of a *complementary* form such that the product of a given form and its complement can be expressed as a determinant for which the closure property is inherent. A number of forms with the closure property are constructed—a number sufficient to indicate that the theory here developed has a reasonable content.

The treatment in general is algebraic though geometric interpretations are constantly helpful. The striking application of the elliptic functions to the Poncelet case seems not to be capable of immediate extension to forms of higher degree since the conditions for closure can not be expressed by means of integrals of the first kind alone.

1. *Poristic Forms and their Complements.*—We begin with the double binary form of order (k, κ)

$$F = (\alpha t)^k (a\tau)^\kappa = 0 \tag{1}$$

and with one pair of values t_0, τ_0 which satisfies it. The value t_0 determines in (1) κ values of τ, say $\tau_0, \tau_1, \cdots$, each of which determines $k - 1$ values of t in addition to t_0 and these values of t determine in turn new values of τ. Proceeding in this way we obtain values $t_0, t_1, t_2, \cdots$; $\tau_0, \tau_1, \tau_2, \cdots$. If after a finite number of repetitions of this procedure we obtain a set of n values t and ν values τ such that each $t(\tau)$ determines in (1) κ (k) values of $\tau(t)$ which are already included in the set of $\nu(n)$ values of $\tau(t)$ then we shall say that F *admits the configuration* $\Delta^{k,\,\kappa}_{n,\,\nu}$, $t_0, t_1, t_2, \cdots, t_{n-1}$; $\tau_0, \tau_1, \tau_2, \cdots, \tau_{\nu-1}$. Indeed in any geometric interpretation of the form F the closed set just mentioned would give rise to a geometric configuration. This configuration would have moreover a type of regularity since each of the t's determines κ of the τ's and each of the τ's is determined by k of the t's. It is clear from the method of derivation of the closed set that the configuration is determined by any one of its *elements* t or of its *elements* τ.

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[*] Prepared under the auspices of the Carnegie Institution of Washington, D. C.

[†] "I polygoni di Poncelet," Torino (1889).

[‡] "Poncelet polygons," *Science* (Feb. 4, 1916), pp. 149–158.

[§] *Proceedings of the National Academy of Sciences*, Vol. 1 (Aug., 1915), p. 464; and Vol. 2 (June, 1916), p. 337.

The number of elements t which with elements τ, i.e., of *pairs* of elements t, τ, will satisfy the form F is $n\kappa$, or equally well νk, so that

$$(2) \qquad n\kappa = \nu k.$$

If F admits one configuration $\Delta^{k,\,\kappa}_{n,\,\nu}$ it does not necessarily admit an infinite number. Classic examples of this are at hand. If however it does admit an infinite number the form F will be called *poristic* and the infinitude of configurations which it determines will be called a *porism*.

When a configuration $\Delta^{k,\,\kappa}_{n,\,\nu}$ satisfies a form F there is determined a *complementary configuration* $\Delta^{n-k,\,\nu-\kappa}_{n,\,\nu}$ made up of those pairs of elements t, τ of the given configuration which do *not* satisfy F.

If the form F is poristic the ∞^1 sets of n elements t which belong to the ∞^1 configurations $\Delta^{k,\,\kappa}_{n,\,\nu}$ are determined by a *pencil* of binary n-ics, $(\gamma t)^n + \lambda(\delta t)^n = 0$. For evidently a set of n elements t is determined by any element of the set. Similarly the ∞^1 sets of ν elements τ in the ∞^1 configurations are determined by a pencil of binary ν-ics, $(c\tau)^\nu + \mu(d\tau)^\nu = 0$. Since each set of n t's makes up with a unique set of ν τ's a configuration and conversely, these two pencils are projectively related, and for proper choice of $(c\tau)^\nu$, $(d\tau)^\nu$ in the second pencil we shall have $\mu = \lambda$. Eliminating λ between the two pencils we get

$$(3) \qquad D_{n,\,\nu} = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix} = 0.$$

The form $D_{n,\,\nu}$ from its nature as a determinant is poristic. It determines ∞^1 *complete* configurations $\Delta^{n,\,\nu}_{n,\,\nu}$, i.e., sets of n t's and ν τ's such that each t determines *all* the τ's and vice versa. These configurations arise from the vanishing of the general column, $\lambda_1(\gamma t)^n + \lambda_2(\delta t)^n = 0$, $\lambda_1(c\tau)^\nu + \lambda_2(d\tau)^\nu = 0$, of the determinant. We shall call the form $D_{n,\,\nu}$ a *complete* poristic form and its configurations a *complete* porism. Clearly $D_{n,\,\nu}$ determined as above will contain the original poristic form as a factor and the complementary factor

$$(4) \qquad F' = (\beta t)^{n-k}(b\tau)^{\nu-\kappa}$$

will also be poristic and will admit the configurations $\Delta^{n-k,\,\nu-\kappa}_{n,\,\nu}$ which are complementary to the configurations $\Delta^{k,\,\kappa}_{n,\,\nu}$ which satisfy F. Hence

(5) *Every poristic double binary form* F *determines a complementary poristic form* F' *such that each configuration of* F *coupled with a complementary configuration of* F' *makes up a configuration of the complete poristic form* $FF' = D_{n,\nu}$.

2. *Conditions for Closure. Corollaries.*—We shall now prove the theorem

(6) *The necessary and sufficient conditions that a given form* $F_{k,\kappa}$ *shall be poristic with configurations* $\Delta_{n,\nu}^{k,\kappa}$ (*where* n, ν, k, κ *are integers subject to the condition* (2)) *are* (A) *that the given form shall admit* ONE *such configuration, and* (B) *that there shall exist a form* $F_{n-k,\nu-\kappa}$ *which shall admit the complementary configuration.*

The necessity of these conditions appears in Sec. 1; we here prove them sufficient. Let the form F in (1) admit the $\Delta_{n,\nu}^{k,\kappa}$ with elements $t_0, t_1, \cdots, t_{n-1}$; $\tau_0, \tau_1, \cdots, \tau_{\nu-1}$. By hypothesis ($B$) the form F' in (4) exists such that in the product

$$FF' = (\alpha t)^k (a\tau)^\kappa \cdot (\beta t)^{n-k} (b\tau)^{\nu-\kappa} = 0$$

each of the n t's determines all of the ν τ's and vice versa. Suppose that for $t = t_0, t_1, \cdots, t_{n-1}$ the product FF' takes the values $r_0(d\tau)^\nu$, $r_1(d\tau)^\nu$, $\cdots$, $r_{n-1}(d\tau)^\nu$ where $(d\tau)^\nu = 0$ has roots $\tau_0, \cdots, \tau_{\nu-1}$. We can determine (in any one of ∞^1 ways) a binary n-ic, $(\gamma t)^n$, which for $t = t_0, \cdots, t_{n-1}$ takes the values $r_0, \cdots, r_{n-1}$. Then the difference $FF' - (\gamma t)^n \cdot (d\tau)^\nu$ vanishes identically for $t = t_0, \cdots, t_{n-1}$ and therefore contains the factor $(\delta t)^n$, where $(\delta t)^n = 0$ has roots $t_0, \cdots, t_{n-1}$, and a complementary factor, say $-(c\tau)^\nu$. Hence

$$FF' = (\gamma t)^n \cdot (d\tau)^\nu - (\delta t)^n \cdot (c\tau)^\nu = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix}.$$

But as has been pointed out the determinant is necessarily poristic and admits ∞^1 configurations $\Delta_{n,\nu}^{n,\nu}$. Since it factors into FF' each configuration $\Delta_{n,\nu}^{n,\nu}$ breaks up into configurations $\Delta_{n,\nu}^{k,\kappa}$ and $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ which satisfy respectively the forms F and F'. Hence each of the forms F and F' are poristic.

Even if the given form F admits one $\Delta_{n,\nu}^{k,\kappa}$ the number, $n(\nu - \kappa)$, of *apparent* conditions on an arbitrary form F' which ensure that it shall admit the complementary $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ is more than F' with $(n-k+1)(\nu-\kappa+1)$ coefficients will in general satisfy. However F' will certainly exist if $(n - k + 1)(\nu - k + 1) \geq n(\nu - \kappa) + 1$ or if

$$n - k \geq (k - 1)(\nu - \kappa). \tag{7}$$

The cases hitherto considered relate to forms F for which $k = \kappa$ and therefore $n = \nu$. Then (7) becomes

$$(2 - k)(n - k) \geq 0. \tag{8}$$

The case $k = \kappa = 1$ presents no interest. But for $k = \kappa = 2$ the relation

(8) is satisfied and the complementary form F' must exist. Thus we find the classic theorem of Poncelet:

(9) *If a form* $F_{2,\,2}$ *admits one configuration* $\Delta^{2,\,2}_{n,\,n}$ *it admits infinitely many and is poristic.*

Moreover it is known that poristic forms $F_{2,\,2}$ with configurations $\Delta^{2,\,2}_{n,\,n}$ actually exist for all values $n \geqq 3$. Hence the complementary poristic forms also exist, i.e.,

(10) *There exist poristic forms* $F_{n\,2,\,n-2}$ *with configurations* $\Delta^{n-2,\,n=2}_{n,\,n}$ *for all values* $n \geqq 4$.

The extreme cases where the orders of F' are as low as possible are worth noting. Thus if $n - k = 0 = \nu - \kappa$ and F' therefore reduces to a constant the relation (7) is satisfied and we can state that

(11) *If a form* $F_{k,\,\kappa}$ *admits one configuration* $\Delta^{k,\,\kappa}_{k,\,\kappa}$ *it is completely poristic.*

If $n - k = 1 = \nu - \kappa$ we find from (2) that $k = \kappa$. Then the complementary form F' is $(\beta t)(b\tau)$ and each of the $k + 1$ t's determines one of the $k + 1$ τ's in the projectivity $F' = 0$. Hence

(12) *If a form* $F_{k,\,k}$ *admits one configuration* $\Delta^{k,\,k}_{k+1,\,k+1}$ *it will be poristic if and only if furthermore the* $k + 1$ *elements* t *are projective to the* $k + 1$ *elements* τ *in the order indicated by the complementary configuration* $\Delta^{1,\,1}_{k+1,\,k+1}$.

This checks the theorem of Meyer which states that if a tetrahedron is inscribed in a cubic curve C with points t and circumscribed to a cubic curve C' with planes τ the curves will not in general admit ∞^1 such tetrahedra. Meyer's condition for the poristic case is that a second such tetrahedron shall exist. But since the incidence condition of point t and plane τ is $F = (\alpha t)^3(a\tau)^3 = 0$, the theorem (12) furnishes as the simpler additional condition for a porism that the four points of the given tetrahedron shall have the same double ratio on C as the four planes have on C'.

It may be observed that in the poristic case of theorem (12) we may always subject τ to such a linear transformation that the complementary form F' becomes $t - \tau$. Then the form F becomes symmetric and vanishes when t, τ belong to the same set of the involution I^{k+1}_1 determined by the pencil, $(\gamma t)^{k+1} + \lambda(\delta t)^{k+1} = 0$. Theorem (12) then reads

(13) *If a symmetric form* $F_{k,\,k}$ *admits one configuration* $\Delta^{k,\,k}_{k+1,\,k+1}$ *it admits* ∞^1 *such configurations.*

3. *Covariant Porisms.*—In the actual construction of poristic forms certain porisms covariantly connected with the given porism are very useful. They are defined as follows. Let the form $F_{k,\,\kappa}$ be poristic with configurations $\Delta^{k,\,\kappa}_{n,\,\nu}$ whence $FF' = D_{n,\,\nu}$. Let S, T, U denote respectively the correspondences from t to τ determined by F, F', $D_{n,\,\nu}$. We use Severi's (Lezioni) definitions for the sum, product, and inverse of these correspondences. Then

$$U = S + T, \qquad U^{-1} = S^{-1} + T^{-1},$$

$$UU^{-1} = (S + T)(S^{-1} + T^{-1}) = SS^{-1} + TT^{-1} + ST^{-1} + TS^{-1}.$$

For these products we may state at once that

(14) *The products* SS^{-1}, TT^{-1}, UU^{-1} *are symmetric correspondences between* t *and* t' *given respectively by forms* $G_{\kappa(k-1,\, \kappa(k-1)}$, $G'_{(\nu-\kappa)(n-k-1),\,(\nu-\kappa)(n-k-1)}$, $I^{\nu}_{n-1,\, n-1}$ *which express that in* $F = 0$, $F' = 0$, $D_{n,\,\nu} = 0$ *respectively* t *and* t' *determine a common* τ. *Here* $I_{n-1,\, n-1} = 0$ *expresses that* t, t' *belong to the same set of the involution* $(\gamma t)^n + \lambda(\delta t)^n = 0$.

(15) *The products* ST^{-1} *and* TS^{-1} *are correspondences between* t, t' *given by forms* $H_{k(\nu-\kappa),\, \kappa(n-k)}$, $H'_{\kappa(n-k),\, k(\nu-\kappa)}$ *which express respectively that* t *in* $F = 0$ *and* t' *in* $F' = 0$, *and* t *in* $F' = 0$ *and* t' *in* $F = 0$, *determine a common* τ. *The forms* H, H' *interchange with* t, t' *since* ST^{-1} *and* TS^{-1} *are inverses.*

Corresponding to the sum of products above we have at once the identity among the forms just defined

$$(16) \qquad I^{\nu} = G\cdot G'\cdot H\cdot H'.$$

Evidently if the forms F, F' are poristic and have the closure property this must be true of the covariant forms I, G, G', H, H' as well and these will be called *covariant porisms.* It is clear moreover that all of these forms can be deduced from F by rational processes whence they are *rational covariants* of F.

On interchanging the order of the factors in the above products and interchanging also n, ν; k, κ; t, τ; t', τ' we have correspondences $S^{-1}S$, $T^{-1}T$, $U^{-1}U$, $S^{-1}T$, $T^{-1}S$ determined by forms $\overline{G}$, $\overline{G'}$, $\overline{I}$, $\overline{H}$, $\overline{H'}$ for which

$$(17) \qquad \overline{I}^{n} = \overline{G}\cdot\overline{G'}\cdot\overline{H}\cdot\overline{H'}.$$

For a complete porism the correspondence T does not exist and $G = I^{\nu}$ but as a rule it will be necessary that the forms I and $\overline{I}$ factor and that various powers of these factors shall be distributed among the forms G, H in (16) and (17). This is true for example of the Poncelet porisms $\Delta^{2,\,2}_{n,\,n}$ $(n > 3)$ for which I is an $(n - 1,\, n - 1)$ form and G a $(2, 2)$ form.

An especially symmetric type of porism may occur when $G = I$ and $\overline{G} = \overline{I}$. Then $\kappa(k - 1) = n - 1$ and $k(\kappa - 1) = \nu - 1$. On eliminating n, ν from (2) we find that $(\kappa - k)(\kappa - 1)(k - 1) = 0$. Since $\kappa > 1$ and $k > 1$ this requires that $\kappa = k$ and therefore $n = \nu$ and $k(k - 1) = n - 1$. In this case the configurations are $\Delta^{r+1,\, r+1}_{r^2+r+1,\, r^2+r+1}$'s $(r = k - 1)$. If $r = p^j$ where p is a prime a grouping of t's and τ's is immediately suggested by the finite plane geometry (mod p^j) where of the totality of $r^2 + r + 1$ points t and $r^2 + r + 1$ lines τ there are $r + 1$ lines τ on a point t and $r + 1$ points t on a line τ. For such configurations we have the theorem:

(18) *If a form* $F_{r+1,\,r+1}$ ($r \geqq 2$) *admits one configuration* $\Delta^{r+1,\,r+1}_{r^2+r+1,\,r^2+r+1}$ *so arranged that any two* t's *determine a common* τ (*or vice versa*) *it admits infinitely many.*

For then the symmetric form G admits a configuration $\Delta^{r^2+r,\,r^2+r}_{r^2+r+1,\,r^2+r+}$ and according to (13) is poristic. Hence F also must have the closure property. This series of cases includes for $r = 2$ that of Professor White.* The question as to whether for values of $r > 2$ F can admit *one* configuration of the sort described in (18) is still open.

4. *The Group Problem of the Porism.*—In order that a form $F_{k,\,\kappa}$ may admit a configuration $\Delta^{k,\,\kappa}_{n,\,\nu}$ it is necessary that the n t's be assorted into ν sets of k each (with which we associate the values τ) in such a way that each t is included in κ of the ν sets. This arrangement into sets must be determined before the conditions of theorem (6) can be applied. The problem of finding such arrangementsbe longs to the theory of finite groups and the solution for given values of n, ν, k, κ is not necessarily unique. Indeed the complications of this tactical problem are the most serious bar to any general discussion of the porisms. Certain series of cases may be treated as a class. Such for example are the Poncelet polygons of n sides, and the checkerboard configurations of n^2 points of Sec. 10. As a rule however each case presents its own peculiarities. Precisely the same tactical problem appears in the formation of irrational (i.e., non-symmetric) invariants of a set of n points in an S_{k-1} of weight ν and degree κ.

If $F_{k,\,\kappa}$ is poristic as above and if we transform its variables by means of the equations

$$t = \frac{(\lambda t')^r}{(\mu t')^r}, \qquad \tau = \frac{(l\tau')^\rho}{(m\tau')^\rho}$$

then the transformed form will likewise have the closure property with configurations $\Delta^{rk,\,\rho\kappa}_{rn,\,\rho\nu}$. The peculiarity of poristic forms which arise by such transformation from simpler forms is that the nr t''s ($\nu\rho$ τ''s) divide into $n(\nu)$ sets of $r(\rho)$ each such that every $t'(\tau')$ in a set determines the same values of $\tau'(t')$. Such a grouping may be described as *imprimitive.* We shall however consider only such configurations for which the elements τ vary with t and vice versa except in the case of the complete porisms.

5. *A Geometric Source of All Poristic Double Forms.*—We shall always choose t so that, in the form $F_{k,\,\kappa}$, $k \leqq \kappa$. We can arrange the terms in $F_{k,\,\kappa}$ according to powers of t as in

$$(19) \qquad t^k(a_0\tau)^\kappa + t^{k-1}(a_1\tau)^\kappa + t^{k-2}(a_2\tau)^\kappa + \cdots + (a_k\tau)^\kappa.$$

Then if t is the parameter of an S_{k-1} u of the rational norm curve N^k in S_k,

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* Cf. Coble, *Proceedings of the National Academy of Sciences*, Vol. 2 (1916), p. 530.

$$u_0 = t^k, \qquad u_1 = t^{k-1}, \qquad u_2 = t^{k-2}, \qquad \cdots, \qquad u_k = 1;$$

and if τ is the parameter of a point x on the rational curve R^κ of order κ in S_k,

$$(20) \qquad x_0 = (a_0\tau)^\kappa, \qquad x_1 = (a_1\tau)^\kappa, \qquad x_2 = (a_2\tau)^\kappa, \qquad \cdots, \qquad x_k = (a_k\tau)^\kappa;$$

it is clear that (19), or $F_{k,\kappa} = 0$, furnishes the incidence condition of point x of R^κ with S_{k-1} u of N^k in S_k. Thus every double form may be interpreted as an incidence condition of the dual elements on two rational curves one of which is a rational norm curve.

If $F_{k,\kappa}$ is poristic the values of t are arranged in closed sets of n determined by the involution I_1^n, $(\gamma t)^n + \lambda(\delta t)^n = 0$. In each set of n we take the $\binom{n}{k}$ sets of k values of t. These determine $\binom{n}{k}$ points in S_k, each point being the meet of the k S_{k-1}'s determined on N^k by the k t's. As the set of n t's runs through I_1^n the $\binom{n}{k}$ points run over the *involution curve of* I_1^n *with reference to* N^k. The order of the involution curve is $\binom{n-1}{k-1}$ since the S_{k-1} of N^k determined by $t = t_1$ meets the curve in those points obtained by taking with $t = t_1$ any $k - 1$ out of the $n - 1$ remaining in the set of I_1^n which contains t_1. If the I_1^n is that associated with a form F, poristic with configurations $\Delta_{n,\,\nu}^{k,\,\kappa}$, then, of the $\binom{n}{k}$ points on the involution curve associated with a set of n t's, ν are isolated by the ν values of τ and lie on the rational curve (20) of order κ, whence the involution curve has the rational constituent R^κ. Conversely if the involution curve of I_1^n with reference to N^k has a rational constituent R^κ the incidence condition of point τ of R^κ with S_{k-1} t of N^k is a poristic form. For the n S_{k-1}'s of N^k determined by a set of I^n each meets the involution curve and therefore also its constituent R^κ only in points such that all the S_{k-1}'s of N^k through these points are found in the set of n whence the incidence condition would necessarily have the closure property. Hence

(21) *Every poristic double form is the incidence condition with a rational norm curve of a rational constituent of an involution curve with reference to the norm curve.*

This theorem is particularly useful in determining poristic forms $F_{2,\,n}$ since the conditions that a plane curve be rational are easy to apply. For the Poncelet polygons of n sides attached to $F_{2,\,2}$, as is well known, the involution curve is composed of $(n - 1)/2$ conics if n is odd and if n is even of $(n - 2)/2$ conics and a line.

6. *The Poristic* $F_{2,\,3}$ *with Configurations* $\Delta_{4,\,6}^{2,\,3}$.—We now proceed to prove the existence of a number of poristic forms of orders greater than (2, 2) and in some cases to effect their construction. For forms $F_{2,\,\kappa}$ with configurations $\Delta_{n,\,\nu}^{2,\,\kappa}$ we name the n t's of a configuration $t_0, t_1, \cdots, t_{n-1}$ and

the ν τ's by τ_{ij} if τ_{ij} determines t_i, t_j in $F_{2,\kappa} = 0$. Then for $F_{2,3}$ with a $\Delta_{4,6}^{2,3}$ we have $4t$'s, t_0, t_1, t_2, t_3 and $6\tau_{ij}$'s $(i, j = 0, \cdots, 3;\ i \neq j)$. It is convenient to think of the t's as the sides and the τ's as the vertices of a complete quadrilateral such that in $F_{2,3} = 0$ the sides are paired with vertices on them. In the complementary configuration $\Delta'^{2,3}_{4,6}$ the sides are paired with vertices not on them. The number of pairs in each configuration is 12 and the complementary forms F, F' have 12 coefficients each so that the requirements of theorem (6) can be satisfied by imposing not more than two conditions on the $4t$'s and 6τ's.

To find these conditions we use the covariant correspondences of Sec. 3. Since $SS^{-1} = I,\ TT^{-1} = I,\ ST^{-1} = TS^{-1} = I^2$, the condition (16) is satisfied without imposing any conditions on $\Delta_{4,6}^{2,3}$. For the inverse correspondences we find however that $\bar{I}$ breaks up into two symmetric correspondences $\bar{J}\bar{K}$ such that $\bar{J}$ is satisfied by a pair τ_{ij}, τ_{ik} of adjacent vertices of the quadrilateral and $\bar{K}$ by a pair τ_{ij}, τ_{kl} of opposite vertices. Then $\bar{G} = \bar{G}' = \bar{J}$, $\bar{H} = \bar{H}' = \bar{J}\bar{K}^2$.

Since τ_{ij}, τ_{kl} satisfy the (1, 1) symmetric form $\bar{K}$ they are partners in an involution whose fixed points we shall take to be $0, \infty$. Then in non-homogeneous coördinates the six τ's are

$$\begin{array}{c} \tau_{01}, \tau_{02}, \tau_{03} \\ \tau_{23}, \tau_{31}, \tau_{12} \end{array} = \begin{array}{c} \lambda, \quad \mu, \quad \nu \\ -\lambda, -\mu, -\nu. \end{array}$$

Hence the three τ's which correspond respectively to t_0, t_1, t_2, t_3 are determined by the cubics $(\tau - \lambda)(\tau - \mu)(\tau - \nu)$, $(\tau - \lambda)(\tau + \mu)(\tau + \nu)$, $(\tau + \lambda)(\tau - \mu)(\tau + \nu)$, $(\tau + \lambda)(\tau + \mu)(\tau - \nu)$. Since for all values of t in $F_{2,3}$ we obtain at most three linearly independent cubics, these four must be connected by a linear relation whose coefficients are $(\mu - \nu)(\nu - \lambda)(\lambda - \mu)$, $(\mu - \nu)(\nu + \lambda)(\lambda + \mu)$, $(\mu + \nu)(\nu - \lambda)(\lambda + \mu)$, $(\mu + \nu)(\nu + \lambda)(\lambda - \mu)$. Denote the cubics in τ with these respective coefficients by C_0, C_1, C_2, C_3 whence

$$C_0 + C_1 + C_2 + C_3 \equiv 0.$$

Let us now determine 4 quadratics in t, Q_0, Q_1, Q_2, Q_3 such that

$$Q_0 + Q_1 + Q_2 + Q_3 \equiv 0,$$

and for $t = t_i$, $Q_i = -3\rho$ and $Q_k = \rho$ $(k \neq i)$. Such quadratics are

$$Q_0 = r_1 + r_2 + r_3, \qquad Q_i = -3r_i + r_j + r_k \quad (i, j, k = 1, 2, 3),$$

$$r_i = (t_0 - t_i)(t - t_j)(t - t_k).$$

Hence in

$$C_0Q_0 + C_1Q_1 + C_2Q_2 + C_3Q_3 \tag{22}$$

we have an $F_{2,3}$ which admits the given $\Delta_{4,6}^{2,3}$. Moreover by changing the sign of τ we get an $F'_{2,\,3}$ which admits the complementary $\Delta'^{2,\,3}_{4,\,6}$. Thus the conditions of theorem (6) are satisfied. The $F_{2,\,3}$ (22) and one of its configurations have the 9 constants, $t_0, \cdots, t_3, \lambda, \mu, \nu, 0, \infty$ whence F alone has 8. Therefore

(23) *The form* $F_{2,\,3}$ *in* (22), *subject to three conditions and containing* 8 *constants or two absolute constants is poristic and admits* ∞^1 *configurations* $\Delta^{2,\,3}_{4,\,6}$. *It is the only type which admits such configurations.*

The degenerate configurations are worth noting. Let as in Sec. 5 t be a line of the norm conic K and τ a point of the rational involution cubic curve C. Suppose that on the conic t_0 and t_1 coincide. Then τ_{01} is a common point of K and C. Moreover now the six τ's are $\left(\begin{smallmatrix} a & b & -b \\ -a & -b & b \end{smallmatrix}\right)$. Thus t_2, t_3 are lines of C since on them two intersections coincide. The pencil of sextics in τ is a pencil of cubics in τ^2 so that it happens four times that $\tau^2 = b^2$ counts twice. Hence the four configurations of the above type account for four of the six common points and the eight common lines of C and K.

In the pencil of sextics there will be either one other sextic with two double roots at $0, \infty$ or two sextics each with one double root at $0, \infty$ respectively. In the pencil of quartics t there are six which have a double root and four of these have been accounted for. If another has only one double root it leads as above to further common lines of C and K. But these have all been accounted for. Hence there is one further quartic with two double roots $t_0 = t_1$, $t_2 = t_3$ and the six τ's are $\left(\begin{smallmatrix} \tau_{01} & 0 & \infty \\ \tau_{23} & 0 & \infty \end{smallmatrix}\right)$. Hence the two tangents t_0, t_2 of K meet in the node $0, \infty$ of C and the points τ_{01}, τ_{23} of C are the two further common points of C and K. Thus C is merely the involution curve of an I_1^4 on a conic K which has acquired a node due to the fact that the I_1^4 contains a perfect square.

7. *The Poristic* $F_{2,\,4}$ *with Configurations* $\Delta^{2,\,4}_{5,\,10}$.—The case treated in the preceding section can be extended one step as indicated in the closing sentence. The involution curve of an I_1^5 on a conic K is a Lüroth quartic curve. If the involution contains three members each with two double roots then the quartic involution curve is rational. If τ_{ij} is a parameter on this rational curve C such that the tangents t_i, t_j of K meet in the point τ_{ij} of C then there exist ∞^1 configurations $\Delta^{2,\,4}_{5,\,10}$ consisting of 5 tangents of K whose 10 meets are on C, and the incidence condition of tangent t_i of K and point τ_{ij} of C, an $F_{2,\,4}$ form, will admit these configurations. The general I_1^5 with 8 constants is here subject to three conditions to account for the three pairs of double roots so that the poristic $F_{2,\,4}$ has five constants or two absolute constants.

If in a set of I_1^5, $t_0 = t_1$, $t_2 = t_3$ then the tangent t_4 of K is a double tangent of C while the contacts of tangents t_0, t_2 of K with K are points of C.

If in a set of I_1^5, $t_0 = t_1$ then the lines t_2, t_3, t_4 are common tangents of C and K and the point t_0 of K is a common point of C and K. The first case happens three times, the second case twice, so that the common points and lines of C and K are accounted for.

The existence of this case implies the existence of the complementary poristic form $F_{3,\,6}$ with configurations $\Delta_{5,\,10}^{3,\,6}$.

The I_1^3 (with Poncelet conics K, C), the I_1^4, and the I_1^5 are the only involutions with a sufficient number of double elements to allow the *complete* involution curve C to acquire a sufficient number of double or multiple points not on the conic K to become rational. Perhaps the series may be continued by allowing multiple points of C to appear on K. We shall however pass on to other configurations for which the involution curve C has a rational part.

8. *The Poristic* $F_{2,\,3}$ *with Configurations* $\Delta_{6,\,9}^{2,\,3}$.—There are but two distinct ways in which six t's, $t_1, \cdots, t_6$ can be arranged in nine pairs in such a manner that each t occurs in three pairs. For both ways the t's divide into two sets of three arranged as in a matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

In the one way each t is paired with a t in the same row or column to form a pair τ_{ij}. For such a configuration the form $F_{2,\,3}$ could not be poristic. In fact the symmetric (3, 3) form G of SS^{-1} which coördinates to the line t_1 of K the lines t_4, t_2, t_3 would break up into a (2,2) form for which t_1 determines t_2, t_3 and a (1, 1) form for which t_1 determines t_4. Thus the rational cubic C determined by G would factor into a conic and a line which would require that the form $F_{2,\,3}$ should factor.

The second grouping which, as we shall see, leads to a poristic $F_{2,\,3}$ coördinates a t in one row to each of the t's in the other row, i.e., t_i ($i = 1, 2, 3$) and t_j ($j = 4, 5, 6$) determine τ_{ij}. If t is a line of a conic K and τ_{ij} a point of the rational cubic C then the 9-point configuration τ_{ij} is cut out on C by two triads of tangents of K. There are on C ∞^4 such 9-point configuration, but of these only ∞^3 for which the six lines touch a conic. The form $F_{2,\,3}$ if poristic will admit ∞^1 of these configurations. Let us study under this hypothesis the necessary behavior of the covariant porisms. The form G symmetric of degree three in t, t' attached to SS^{-1} will for $t = t_1$, or t_2, or t_3 determine the triad $t' = t_4$, t_5, t_6; and for $t = t_4$, or t_5, or t_6 the triad $t' = t_1$, t_2, t_3. Hence as the $\Delta_{6,\,9}^{2,\,3}$'s more around we have a pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ such that this member of the pencil determines involutorily another member $(\gamma t)^3 - \lambda(\delta t)^3$. This can be visualized by taking sections of a space cubic curve by planes on a line where the planes π_1, π_2 cut out the fixed members $(\gamma t)^3$, $(\delta t)^3$ of the above involution. Then points t, t' on the space cubic are apolar to the pair of planes π_1, π_2 if

$$G = (\gamma t)^3(\delta t')^3 + (\gamma t')^3(\delta t)^3 = 0.$$

Interpreting this form, which is evidently poristic, on K, the point t, t' runs over a cubic C of genus one. The form G has three absolute constants which arise from π_2 after the space cubic and π_1 have been chosen. Hence the cubic C and conic K together have eleven constants, or given C there are ∞^2 conics K each with $\infty^1\Delta_{6,\,9}^{2,\,3}$'s. Thus the ∞^3 configurations whose six lines touch a conic accounted for and it is clear that the existence of one such configuration of six t's for a symmetric (3, 3) form implies the existence of infinitely many.

The cubic C must acquire a node in order that its points may be named by a parameter τ_{ij}. This requires that in one set of six t's, $t_1 = t_2$ and $t_4 = t_5$, or that π_1, π_2 be a pair of planes apolar to two tangent planes of the space cubic on the common line of π_1, π_2, which is one condition. Hence

(24) *There exist poristic forms* $F_{2,\,3}$ *with configurations* $\Delta_{6,\,9}^{2,\,3}$ *depending upon two absolute constants. The existence of one such configuration implies the porism.*

For given F and one configuration Δ the equation of the rational cubic C referred to K is a symmetric (3, 3) form G which admits one $\Delta_{3,\,3}^{3,\,3}$ and therefore according to (11) is poristic. This requires in turn that F be poristic.

In the pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ there are four members with a double root but two of these are paired at the double point so that we have one case

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_4 & t_6 \end{pmatrix}.$$

The double point is either $\tau_{14} = \tau_{25}$ or $\tau_{15} = \tau_{24}$ with further coincidences $\tau_{16} = \tau_{26}$ and $\tau_{34} = \tau_{35}$ so that the lines t_3, t_6 are common tangents of C and K. There remain two cases

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

Then $\tau_{14} = \tau_{24}$, $\tau_{15} = \tau_{25}$, $\tau_{16} = \tau_{26}$ and the lines t_4, t_5, t_6 account for the remaining common tangents of C and K. Moreover corresponding to the double cubics of the involution, $(\gamma t)^3$ and $(\delta t)^3$, there are two cases

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Then $\tau_{15} = \tau_{24}$, $\tau_{16} = \tau_{34}$, $\tau_{26} = \tau_{35}$ and the points t_1, t_2, t_3 of K are on C. Thus the common points of C and K are located. Moreover the pencil of

9-ics τ will have sixteen members with double roots and all of these appear above.

The configuration $\Delta^{2,3}_{6,9}$ is a special case of the *checkerboard configuration* of n^2 points which are the intersections of one set of n lines with another set of n lines. These are discussed further in the next two sections.

9. *The Poristic $F_{2,4}$ with Checkerboard Configurations $\Delta^{2,4}_{8,16}$.*—If a quartic Q has an inscribed checkerboard cut out by the two sets of four lines Q_1, Q_2 then Q has the form $Q_1 + \mu Q_2 = 0$. This form contains $2 \times 8 + 1$ or 17 constants whereas Q has but 14 so that a given quartic has presumably ∞^3 checkerboards. We should expect therefore a finite number for which the eight lines touch a conic K. But as a matter of fact if one set of eight lines touches K there will be ∞^1 sets which touch K. For if $(\gamma t)^4$ and $(\delta t)^4$ represent these sets of four lines on K, the equation of Q in Darboux coördinates t, t' is $(\gamma t)^4(\delta t')^4 + (\gamma t')^4(\delta t)^4 = 0$ and this form is poristic. This pair of binary quartics on K has five absolute constants whence Q is subject to one invariant condition. Thus we find the theorem analogous to the Lüroth theorem on the inscribed pentagons of a quartic curve:

(25) *If a quartic Q passes through the sixteen points of one checkerboard whose eight lines touch a conic K then Q is subject to one invariant condition and has ∞^1 inscribed checkerboards whose lines envelop K.*

The two sets of four lines which make up the checkerboard are pairs of sets of an I_1^4 on K which are involutorily related. If Q should have three double points and be rational with parameter τ then the condition that point τ of Q and line t of K be incident is a poristic $F_{2,4}$. If a binary quartic of the pencil has a double root while the paired quartic has not, i.e., if the pair of quartics is $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \beta & \gamma & \delta \end{smallmatrix}\right)$ then $\alpha, \beta, \gamma, \delta$ are common tangents of K and Q. If both quartics of the pair have a double root as in $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \alpha & \beta & \gamma \end{smallmatrix}\right)$ then the point (a, α) is a double point of Q and the lines b, c, β, γ are common tangents. If both quartics of the pair have a triple root as in $\left(\begin{smallmatrix} a & a & a & b \\ \alpha & \alpha & \alpha & \beta \end{smallmatrix}\right)$ then the point (a, α) is a triple point of Q while b, β are common tangents of K and Q which are inflexional for the latter.

If then the case $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \alpha & \beta & \gamma \end{smallmatrix}\right)$ occurs three times Q has three nodes and twelve tangents in common with K. All the multiple roots of the pencil are accounted for but clearly the six quartics of the pencil with double roots must have their parameters in involution in order that the fixed pair $(\gamma t)^4$, $(\delta t)^4$ may exist. Such a pencil has only two absolute constants.

If the case $\left(\begin{smallmatrix} a & a & a & b \\ \alpha & \alpha & \alpha & \beta \end{smallmatrix}\right)$ occurs there will remain two cases $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \beta & \gamma & \delta \end{smallmatrix}\right)$ so that Q has a triple point and $2.4 + 2.2$ tangents in common with K. Then $(\gamma t)^4$, $(\delta t)^4$ is any one of the ∞^1 pairs of quartics whose parameters are harmonic to the two with triple points so that for this case also there are two absolute constants.

(26) *There are two types, each with two absolute constants, of poristic $F_{2,\,4}$'s with configurations $\Delta^{2;\,4}_{8;\,16}$ of the checkerboard type. The checkerboards are inscribed in a quartic which for the one type has three nodes; for the other, a triple point.*

10. *The Poristic $F_{2,\,n}$ with Checkerboard Configurations $\Delta^{2;\,n}_{2n,\,n^2}$.*—Let us generalize the configuration of the preceding section to $2n$ lines and n^2 points and inquire as to the existence of porisms with these configurations. As before they must be associated with two particular members of a pencil of binary n-ics if the $2n$ lines envelop a conic K. An n-ic curve and one inscribed checkerboard has $4n + 1$ constants. It is $2n - 5$ conditions that the $2n$ lines envelop K so that then the curve and configuration have $2n + 6$ constants or $2(n - 1)$ absolute constants. However it is then determined by two fixed members of a pencil of binary n-ics which have $2(n - 1) - 1$ absolute constants. Hence

(27) *If a curve C of order n has an inscribed checkerboard of n^2 points and $2n$ lines which envelop a conic K then there are ∞^1 such checkerboards inscribed in C and circumscribed to K. The curve C has $2n - 3$ absolute constants.*

Let us now attempt to make C rational by giving it nodes. Suppose that one member of the pencil of n-ics has k double roots and that the paired members has l double roots. Then each double root of the one member with each double root of the other member determines a node of C so that this pair of n-ics contributes $k + l$ of the $2(n - 1)$ double roots of the pencil and contributes kl nodes to C. If C is rational for r such cases we have

$$
\begin{aligned}
&k_1 + l_1 + k_2 + l_2 + \cdots + k_r + l_r = 2(n - 1), \\
&\qquad k_1l_1 + k_2l_2 + \cdots + k_rl_r = (n - 1)(n - 2)/2.
\end{aligned}
\tag{28}
$$

In each pair of this sort the $n - 2k$ simple roots of the first member will each contribute an l-fold common tangent of C and K. Since it is a consequence of (28) that

$$
(n - 2k_1)l_1 + (n - 2l_1)k_1 + \cdots + (n - 2k_r)l_r + (n - 2l_r)k_r = 4(n - 1),
$$

all the common tangents of C and K are found.

It is however $k - 1$ conditions on the pencil that a member shall have k double roots, $l - 1$ conditions that another shall have l double roots, and one condition that these be paired. Hence the number of absolute constants which remain is

$$
2n - 3 - \sum_{i=1}^{r} (k_i + l_i - 1) = 2n - 3 - \{2(n - 1) - r\} = r - 1.
$$

An evident inequality for the integers k, l is

$$
0 \overline{<} k_i \overline{<} n/2, \qquad 0 \overline{<} l_\iota \overline{<} n/2. \tag{29}
$$

One solution of the diophantine system (28), (29) is

$$(30)\quad n = 2j + 1\begin{cases} k_1, l_1 = j, j \\ k_2, l_2 = j, j - 1; \\ k_3, l_3 = 1, 0 \end{cases} \qquad n = 2j\begin{cases} k_1, l_1 = j, j - 1 \\ k_2, l_2 = j - 1, j - 1 \\ k_3, l_3 = 1, 0. \end{cases}$$

Here $r - 1 = 2$ so that these forms have two absolute constants. But there may well be other solutions of the system.

If the diophantine system above be extended to take account of the possibility of points of multiplicity as great as $n - 1$ on C, the system becomes much more complicated but solutions of this higher type exist. For example construct the pencil of n-ics by taking an n-ic with $(n - 1)$-fold root a and simple root b and another with $(n - 1)$-fold root α and simple root β (one absolute constant) and choose for $(\gamma t)^n$, $(\delta t)^n$ a pair of n-ics of the pencil whose parameters are apolar to the two with $(n - 1)$-fold roots (a second absolute constant). Then C has an $(n - 1)$-fold point at (a, α) and the tangents b, β of K have $(n - 1)$-point contact with C. Hence we have the generalized form of (26).

(31) *There will exist poristic forms* $F_{2,\,n}$ *with configurations* $\Delta^{2,\,n}_{2n,\,n^2}$ *of the checkerboard type whose* $2n$ *lines* t *envelop a conic* K *while the* n^2 *points* τ *describe a rational* n*-ic curve* C *whose multiple points may consist only of double points as in* (30), *or of a single* $(n - 1)$*-fold point, or of points with intermediate multiplicities. The extreme cases actually constructed above both involve two absolute constants.*

11. *The Poristic* $F_{2,\,n-1}$ *with Configurations* $\Delta^{2,\,n-1}_{2n,\,n(n-1)}$.—If in the checkerboard of $2n$ lines and n^2 points a diagonal be selected this implies an involutory mutual ordering of the two sets of n lines. Thus if we name the lines $\binom{t_1 t_2 \cdots t_n}{s_1 s_2 \cdots s_n}$ and the n^2 points τ_{ij} $(i, j = 1, \cdots, n)$ we may select as diagonal points the n points τ_{ii}. If this diagonal be deleted the resulting configuration will have $2n$ lines as before but only $n^2 - n$ points τ_{ij} $(i \neq j)$. If ∞^1 such configurations were circumscribed to K there would necessarily exist on K an involution I which may be taken as $t' = -t$. Then the form $(\delta t)^n$ of the preceding section would arise from $(\gamma t)^n$ by changing the sign of t. The set of $2n$ lines would have parameters $\binom{t_1 \quad t_2 \cdots \quad t_n}{-t_1 \, -t_2 \cdots -t_n}$. From the curve C there would factor the line joining the fixed points $0, \infty$ of the involution I on K. Of the various rational curves C referred to in (31) let us consider only the one with an $(n - 1)$-fold point determined by $\binom{a,\ \ a,\ \cdots,\ \ a,\ \ b}{-a,\, -a,\, \cdots,\, -a,\, -b}$ for which $(\gamma t)^n = (t - a)^{n-1}(t - b)$, $(\delta t)^n = (t + a)^n(t + b)$. After factoring out the line of I there remains a curve C' of order $n - 1$ with an $(n - 2)$-fold point $(a, -a)$. In the equation of C' we have the one absolute constant implied in $0, \infty, a, b$. If τ_{ij} is a parameter on the rational curve C' we find from the incidence condition of C' and K that

(32) *There exist poristic forms $F_{2,\, n-1}$, involving at least one absolute constant, with configurations $\Delta^{2,\, n-1}_{2n,\, n(n-1)}$ of the type t_i, s_j and τ_{ij} determined by t_i, s_j $(i, j = 1, \cdots, n;\ i \neq j)$.*

12. *The Poristic $F_{3,\, 4}$ with Box Configurations $\Delta^{3,\, 4}_{6,\, 8}$.*—As an instance of poristic double forms with distinct orders both greater than two we take the $F_{3,\, 4}$. Unless this is completely poristic with configurations $\Delta^{3,\, 4}_{3,\, 4}$, the simplest configurations it can admit are $\Delta^{3,\, 4}_{6,\, 8}$'s. For these a grouping immediately at hand is the six faces t and eight vertices τ of a cube or box where $F_{3,\, 4} = 0$ coördinates to each vertex the three faces on it and to each face the four vertices on it. If we call the three pairs of opposite faces $t_1, t_2;\ t_3, t_4;\ t_5, t_6$, the vertices may be named $\tau_{135}, \tau_{246};\ \tau_{136}, \tau_{245};\ \tau_{145}, \tau_{236};\ \tau_{146}, \tau_{235}$ where τ_{ijk} is the vertex on faces t_i, t_j, t_k.

If $F_{3,\, 4}$ is poristic the correspondence SS^{-1} associates $t_3^2, t_4^2, t_5^2, t_6^2$ to t_1. Hence the symmetric form G is a perfect square and $\sqrt{G}$ is a rational symmetric (4, 4) correspondence. In UU^{-1} t_1 corresponds to t_2, t_3, t_4, t_5, t_6 whence the form I factors into $J\sqrt{G}$ where J is a symmetric (1, 1) correspondence or involution whose pairs are opposite faces. Let this involution be $t' = -t$ so that the six faces may be taken as $\pm l, \pm m, \pm n$.

The correspondence $S^{-1}S$ makes τ_{135} correspond to $\tau_{136}^2\tau_{145}^2\tau_{235}^2\tau_{146}\tau_{236}\tau_{245}$ whence the form G factors into K^2L where K, L are symmetric (3, 3) forms which coördinate vertices joined by an edge or by a face diagonal respectively. Moreover $U^{-1}U$ makes τ_{135} correspond to $\tau_{246};\ \tau_{136}, \tau_{145}, \tau_{235};\ \tau_{146}, \tau_{236}, \tau_{245}$ or $\bar{I} = MKL$ where $M = 0$ is the involutory correspondence between opposite vertices. Let us take M to be $\tau' = -\tau$ whence the eight τ's may be taken as $\pm\alpha, \pm\beta, \pm\gamma, \pm\delta$ and the form F must be unaltered by the simultaneous change of sign of t and τ.

In the configuration thus simplified the form F must coördinate the faces and vertices as follows:

$$l:\ -\alpha,\ \beta,\ -\gamma,\ \delta;\qquad m:\ \alpha,\ \beta,\ -\gamma,\ -\delta;\qquad n:\ -\alpha,\ \beta,\ \gamma,\ -\delta$$

$$-l:\ \alpha,\ -\beta,\ \gamma,\ -\delta;\qquad -m:\ -\alpha,\ -\beta,\ \gamma,\ \delta;\qquad -n:\ \alpha,\ -\beta,\ -\gamma,\ \delta.$$

In order to be unaltered by changing the signs of t, τ and to have the solutions indicated for $t = \pm l, \pm m$, $F_{3,\, 4}$ must be

$$\begin{aligned} & r\{(\tau+\alpha)(\tau-\beta)(\tau+\gamma)(\tau-\delta)/(t-l) \\ & \qquad\qquad - (\tau-\alpha)(\tau+\beta)(\tau-\gamma)(\tau+d)/(t+l)\} \\ & + s\{(\tau-\alpha)(\tau-\beta)(\tau+\gamma)(\tau+\delta)/(t-m) \\ & \qquad\qquad - (\tau+\alpha)(\tau+\beta)(\tau-\gamma)(\tau-\delta)/(t+m)\} = 0. \end{aligned} \tag{33}$$

In addition for $t = n$ $F_{3,4}$ must vanish for $\tau = -\alpha, \beta, \gamma, -\delta$ and then will have the proper solutions for $t = -n$. These conditions lead to the equations

$$\frac{r}{s} = \frac{(n+l)}{(n-m)} \frac{(\beta+\alpha)}{(\beta-\alpha)} \frac{(\gamma-\alpha)}{(\gamma+\alpha)} = -\frac{(n+l)}{(n+m)} \frac{(\beta+\alpha)}{(\beta-\alpha)} \frac{(\beta-\delta)}{(\beta+\delta)}$$

$$= -\frac{(n-l)}{(n-m)} \frac{(\gamma-\alpha)}{(\gamma+\alpha)} \frac{(\gamma+\delta)}{(\gamma-\delta)} = \frac{(n-l)}{(n+m)} \frac{(\beta-\delta)}{(\beta+\delta)} \frac{(\gamma+\delta)}{(\gamma-\delta)},$$

which are satisfied by

$$(34) \qquad \begin{aligned} l : m : n &= \frac{1}{(\gamma\alpha-\beta\delta)} : \frac{1}{(\gamma\delta-\alpha\beta)} : \frac{1}{(\alpha\delta-\beta\gamma)}; \\ \frac{r}{s} &= \frac{(\alpha+\beta)}{(\alpha+\gamma)} \frac{(\gamma+\delta)}{(\beta+\delta)} \frac{(\alpha\beta-\gamma\delta)}{(\alpha\gamma-\beta\delta)}. \end{aligned}$$

Hence the form $F_{3,4}$ determined in (33), (34) admits one $\Delta^{3,4}_{6,8}$. Moreover the form $F'_{3,4}$ which arises from $F_{3,4}$ by the change of sign of t alone admits the complementary configuration whence according to (6) the form F is poristic. Two constants for t, τ each have been absorbed in the choice of the involutions $t' = -t$, and $\tau' = -\tau$. By proper choice of the unit points two of the four constants $\alpha, \beta, \gamma, \delta$ can be absorbed leaving two absolute constants for the form $F_{3,4}$ and one configuration, or one absolute constant for the form.

(35) *There exists a poristic form* $F_{3,4}$ *involving one absolute constant with configurations* $\Delta^{3,4}_{6,8}$ *of the box type; or there is in space a system of* ∞^1 *boxes whose faces envelop a space cubic curve and whose corners run over a rational quartic curve.*

With regard to this space quartic curve we remark that its sections and therefore their common apolar quartic all admit the involution $\tau' = -\tau$ and are therefore harmonic. The quartic therefore has a node and no absolute constant. Hence the constant in $F_{3,4}$ must be due to the choice of the series of boxes inscribed in the quartic and each of the ∞^1 series determines its own cubic curve. This situation perhaps deserves further study.

It is apparent that the box is the simplest instance in space of a "cellular configuration" (with one cell), the extension to space of the checkerboard configuration in the plane. It would be interesting to know whether forms F_{3,n^2} with cellular configurations Δ^{3,n^2}_{3n,n^3} containing $(n-1)^3$ cells occur. We shall not however attempt to multiply further examples of poristic forms but will close with a summary of the poristic cases which have been established thus far.

13. *Summary of Poristic Double Forms.*

	Configuration	*Group type*	*Reference*
1°	$\Delta_{n,\,n}^{2,\,2}$	Poncelet polygon	(9)
2°	$\Delta_{n,\,n}^{n-2,\,n-2}$	Complement of 1°	(10)
3°	$\Delta_{k,\,\kappa}^{k,\,\kappa}$	Complete porism	(11)
4°	$\Delta_{k+1,\,k+1}^{k,\,k}$	Transform of I_1^{k+1}	(12,) (13)
5°	$\Delta_{7,\,7}^{3,\,3}$	White's porism	(18)
6°	$\Delta_{7,\,7}^{4,\,4}$	Complement of 5°	
7°	$\Delta_{4,\,6}^{2,\,3}$	Quadrilateral porism	(23)
8°	$\Delta_{5,\,10}^{2,\,4}$	Pentagonal porism	Sec. 7
9°	$\Delta_{5,\,10}^{3,\,6}$	Complement of 8°	
10°	$\Delta_{2n,\,n^2}^{2,\,n}$	Checkerboard	(24), (26), (31)
11°	$\Delta_{2n,\,n(n-1)}^{2,\,n-1}$	Checkerboard without diagonal	(32)
12°	$\Delta_{2n,\,n^2}^{2(n-1),\,n(n-1)}$	Complement of 10°	
13°	$\Delta_{2n,\,n(n-1)}^{2(n-1),\,(n-1)^2}$	Complement of 11°	
14°	$\Delta_{6,\,8}^{3,\,4}$	Box porism	(35)

14. *Extensions to Multiple Forms.*—The double binary forms may be extended in various ways. One natural extension of the (2, 2) incidence condition of line t and point τ of two conics is to the incidence condition of plane and point of two quadrics. We should then inquire as to the existence of polyhedra circumscribed to the one quadric and inscribed in the other. However on naming the planes of one quadric by the parameters (of its generators) λ, μ, and the points of the other by λ', μ' the incidence condition appears in the form

$$(a\lambda)(b\mu)(c\lambda')(d\mu') = 0$$

in which unfortunately the four variables are not on a par.

More interesting, at least from the algebraic point of view, are the triple binary forms,

$$(a\lambda)^l(b\mu)^m(c\nu)^n = 0. \tag{36}$$

Let λ_0, μ_0, ν_0 be any solution of (36). Then given λ_0, μ_0 we obtain $n - 1$ values of ν in addition to ν_0 which satisfy (36); and from any one of these n solutions new solutions with different λ or with different μ are obtained. If after a time no *new* solutions can be obtained by this process from those already known then we shall have a triple form with the *closure* property.

Two types of triple forms with the closure property are suggested by earlier cases. The first is an extension of the form G of Sec. 8. Let the parameters λ, λ', λ'' which determine n-ics in the pencils

$$(\gamma t)^n + \lambda(\delta t)^n = 0, \qquad (\gamma t')^n + \lambda'(\delta t')^n = 0, \qquad (\gamma t'')^n + \lambda''(\delta t'')^n = 0$$

be connected by the involutory relation

$$a_0\lambda\lambda'\lambda'' + a_1(\lambda'\lambda'' + \lambda''\lambda + \lambda\lambda') + a_2(\lambda + \lambda' + \lambda'') + a_3 = 0.$$

On replacing the λ, λ', λ'' by their values in terms of t, t', t'' we obtain a triple (n, n, n) form with the closure property.

Another triple form with the closure property is the determinant

$$\Delta = \begin{vmatrix} (a_0\lambda)^l & (a_1\lambda)^l & (a_2\lambda)^l \\ (b_0\mu)^m & (b_1\mu)^m & (b_2\mu)^m \\ (c_0\nu)^n & (c_1\nu)^n & (c_2\nu)^n \end{vmatrix} = 0 \tag{37}$$

obtained by eliminating the line coördinates ξ_0, ξ_1, ξ_2 in S_2 from the equations

$$\begin{aligned} (a_0\lambda)^l\xi_0 + (a_1\lambda)^l\xi_1 + (a_2\lambda)^l\xi_2 &= 0, \\ (b_0\mu)^m\xi_0 + (b_1\mu)^m\xi_1 + (b_2\mu)^m\xi_2 &= 0, \\ (c_0\nu)^n\xi_0 + (c_1\nu)^n\xi_1 + (c_2\nu)^n\xi_2 &= 0, \end{aligned} \tag{38}$$

which represent rational curves in S_2 of orders l, m, n and parameters λ, μ, ν respectively. The condition $\Delta = 0$ expresses that points λ, μ, ν respectively on these curves are collinear and obviously Δ has the closure property.

If Δ factors then each factor will likewise have the closure property. Special cases of such factorization are furnished by the involution determined by the rational plane curve of order k, $x_i = (d_it)^k$ $(i = 0, 1, 2)$. The condition that points t_0, t_1, t_2 of this curve lie on a line is the vanishing of the determinant

$$|(d_it_j)^k| = (t_0t_1)(t_0t_2)(t_1t_2)\cdot(\alpha_0t_0)^{k-2}(\alpha_1t_1)^{k-2}(\alpha_2t_2)^{k-2}.$$

Here the symmetric $(k-2, k-2, k-2)$ form has the closure property. Similarly if in Δ, $(b_i\mu)^m_{\mu=\nu} = (c_i\nu)^n$, then the factor $(\mu\nu)$ appears and the remaining factor is an $(l, m-1, m-1)$ form symmetric in μ, ν with the closure property. It is clear however from a discussion of these special cases* that the existence of one closed set is not in general sufficient to ensure closure.

If in the form (36) λ, μ, ν be regarded as parameters in three plane pencils on lines L, M, N respectively in S_3, and λ, μ, ν as the coördinates of the intersection of the three planes, then (36) is the equation of a surface of order $l + m + n$ with the lines L, M, N as l-, m-, n-fold lines respectively. If the form has the closure property then the points of the surface divide into closed sets such that the bisecants from any point of the set across any two of the lines L, M, N meet the surface again in points of the set. A striking case of this sort is that of the (2, 2, 2) form for which the bisecant construction for lines M, N leads to an involutory birational transformation I_L of the sextic surface into itself. The closure case occurs when the three involutions I_L, I_M, I_N generate a finite group.

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* Cf. Coble, "Symmetric Binary Forms and Involutions," this JOURNAL, Vol. 32 (1910), p. 333; in particular (104) p. 336.

Three instances of such sextic surfaces may be given. If first in (37) $l = m = n = 2$ the involutions I_L, I_M, I_N generate an abelian G_8. The closed sets of eight points are cut out by three pairs of planes on L, M, N. If second in (37) $l = 2$, $m = n = 3$, and $(b_i\mu)^3_{\mu=\nu} = (c_i\nu)^3$ then the quadric $(\mu\nu)$ factors out leaving a sextic surface with closed sets of 12 points. Now I_μ and I_ν generate a dihedral G_6 and I_L, interchangeable with I_M and I_N generates with G_6 a dihedral G_{12}. If thirdly in (37) $l = m = n = 4$, $(c_i\nu)^4_{\nu=\lambda} = (a_i\lambda)^4$ and, $(b_i\mu)_{\mu=\lambda} = (a_i\lambda)^4$ then the sextic $(\mu\nu)(\nu\lambda)(\lambda\mu)$ factors out leaving a sextic surface with closed sets of 24 points. The involutions now generate a G_{24} isomorphic with the symmetric G_{24}.

These fairly evident cases where factorization of Δ is possible can be generalized to higher orders or to larger numbers of variables. Though I have not been able to find other types of factorization it would seem to be quite likely that such new types exist and their determination presents a fascinating problem.

URBANA, ILL.,
May, 1920.